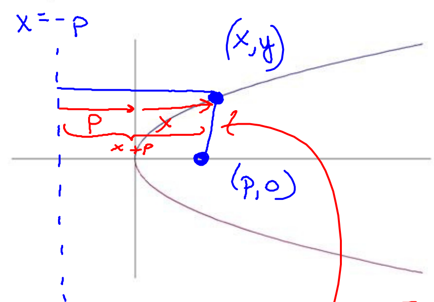
§21.4—The Parabola

Definition: A **parabola** is the set of points equidistant from a given line (called the **directrix**) and a given fixed point (called the **focus**). The point midway between the focus and directrix is called the **vertex**, and the line perpendicular to the directrix that goes through the vertex is called the **axis** (or axis of symmetry).

Parabolas With Their Vertex at the Origin

I. If the focus of a parabola is $(p,0)$ and the directrix is the line $x=-p$, then the vertex is at the origin, and the axis is the x-axis. The **standard form** for the equation of such a parabola is $y^2=4px$. Let's see why.



$$\sqrt{(x-p)^2+(y-0)^2}$$

$$(x+p)^2 = \left(\sqrt{(x-p)^2+y^2}\right)^2$$

$$x^2+2px+p^2 = (x-p)^2+y^2$$

$$\cancel{x^2}+2px+\cancel{p^2} = \cancel{x^2}-2px+\cancel{p^2}+y^2$$

$$y^2=4px$$

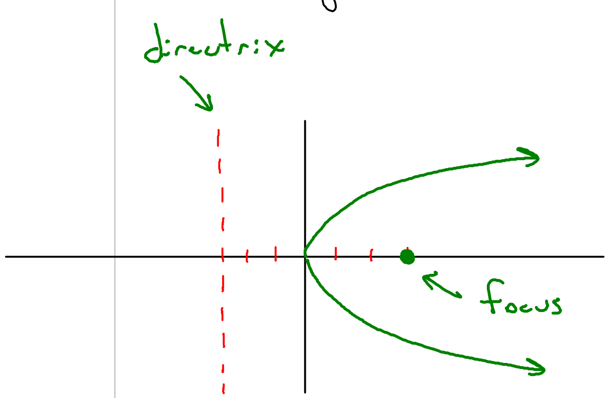
Example 1. Find the focus and directrix of the parabola $y^2=12x$. Then sketch it.

$$y^2=12x \implies 12x = 4px$$

$$12=4p$$

$$p=3 \implies \text{focus: } (3,0)$$

$$\text{directrix: } x=-3$$



Example 2. Find the focus and directrix of the parabola $y^2 = -8x$. Then sketch it.

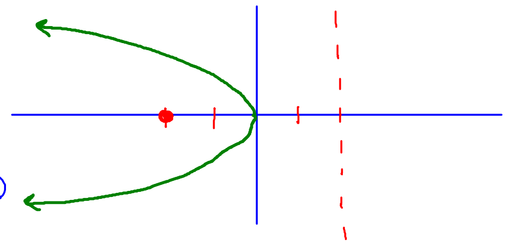
$y^2 = -8x$

$-8\cancel{x} = 4p\cancel{x}$

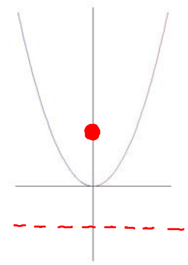
$p = -2$

focus: $(-2,0)$

directrix: $x = 2$



II. If the focus of a parabola is $(0, p)$ and the directrix is $y = -p$, then the vertex is at the origin and the axis is the y-axis. The standard form for the equation of such a parabola is $x^2 = 4py$.



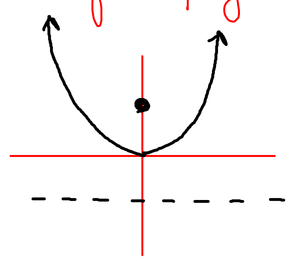
$x^2 = 4py$

Example 3. Find the focus and directrix of the parabola $x^2 = 4y$. Then sketch it.

$x^2 = 4y \Rightarrow 4y = 4py \Rightarrow p = 1$

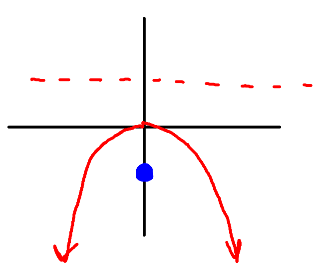
focus: $(0,1)$

directrix: $y = -1$



Example 4. Find the focus and directrix of the parabola $2x^2 = -9y$. Then sketch it.

$\frac{2x^2}{2} = \frac{-9y}{2} \Rightarrow x^2 = -\frac{9}{2}y \Rightarrow 4p = -\frac{9}{2} \Rightarrow p = -\frac{9}{8}$



focus: $\left(0, -\frac{9}{8}\right)$

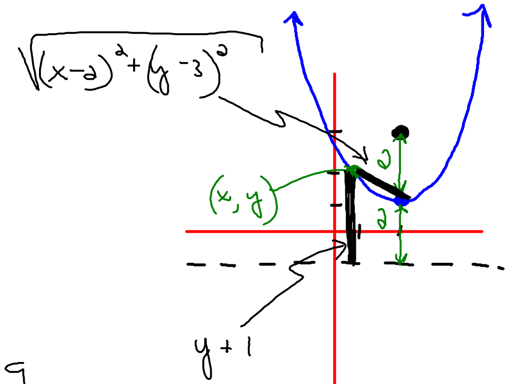
directrix: $y = \frac{9}{8}$

Observation. We can tell we're looking at the equation of a parabola if one variable is squared and the other isn't. The parabola always opens about the non-squared variable.

Example 5. (*vertex not at the origin*) Use the definition of the parabola to find the equation of the parabola with focus $(2,3)$ and directrix $y=-1$.

focus: $(2,3)$

directrix: $y=-1$ ⟵ vertically oriented



$$(y+1)^2=\left(\sqrt{(x-2)^2+(y-3)^2}\right)^2$$

$$y^2+2y+1=(x-2)^2+(y-3)^2$$

$$\cancel{y^2}+2y+1=x^2-4x+4+\cancel{y^2}-6y+9$$

$$2y+1=x^2-4x-6y+13$$

$$\boxed{0=x^2-4x-8y+12}$$

either is fine

$$\text{or}\quad 8y=x^2-4x+12 \Longrightarrow \boxed{y=\frac{1}{8}x^2-\frac{1}{2}x+\frac{3}{2}}$$

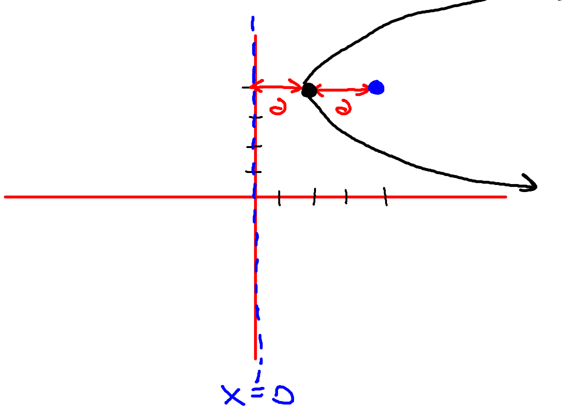
Translation of Axes

So far, we have only looked at parabolas whose vertices are at the origin. A parabola with vertex (h,k) has an equation of the form

$(y-k)^2 = 4p(x-h)$ (axis parallel to the x-axis)

$(x-h)^2 = 4p(y-k)$ (axis parallel to the y-axis)

Example 6. Find the equation of the parabola with vertex $(\overset{h}{2},\overset{k}{4})$ and focus $(4,4)$.



$$(y-k)^2 = 4p(x-h)$$

$$(y-4)^2 = 4p(x-2)$$

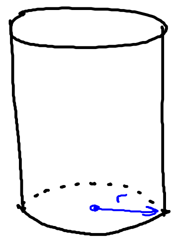
$$p = 2$$

$$(y-4)^2 = 4 \cdot 2(x-2)$$

$$\boxed{(y-4)^2 = 8(x-2)}$$

Example 7. Cylindrical glass beakers are to be made with a height of 3 inches. Express the surface area in terms of the radius of the base and sketch the curve.

no top



h=3

$$SA = \pi r^2 + 2\pi r h$$

$$SA = \pi r^2 + 6\pi r$$

complete the square to get it into standard form

$$\frac{y}{\pi} = \frac{\cancel{\pi} r^2}{\cancel{\pi}} + \frac{6\cancel{\pi} r}{\cancel{\pi}}$$

$$\left(\frac{6}{2}\right)^2 + \frac{y}{\pi} = r^2 + 6\pi + \left(\frac{6}{2}\right)^2$$

$$9 + \frac{y}{\pi} = r^2 + 6\pi + 9$$

$$9 + \frac{y}{\pi} = (r+3)^2 \qquad \boxed{(x-h)^2 = 4p(y-k)}$$

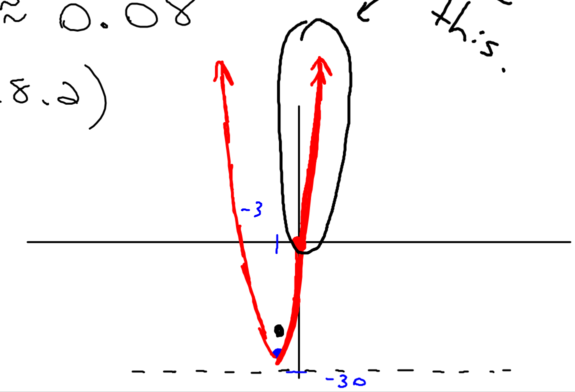
$$\frac{1}{\pi}(y + 9\pi) = (r+3)^2$$

Vertex: $(-3, -9\pi)$

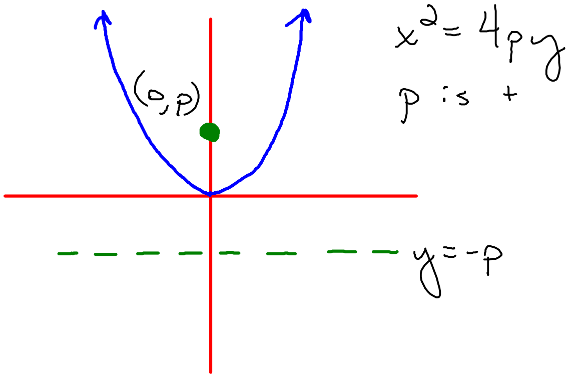
$$4p = \frac{1}{\pi} \Longrightarrow p = \frac{1}{4\pi} \approx 0.08$$

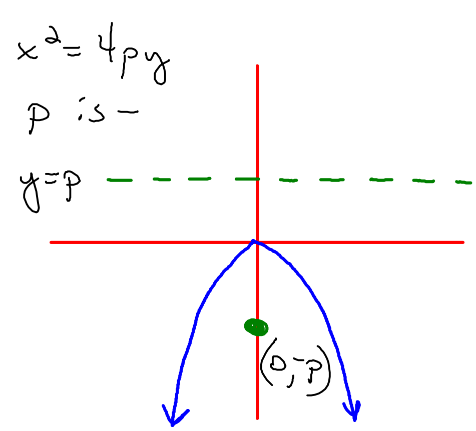
focus: $(-3, -28.2)$

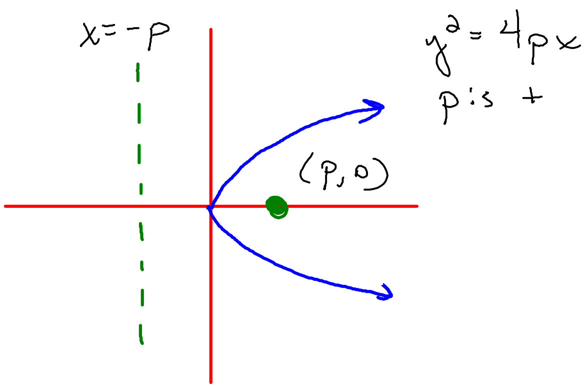
use this.

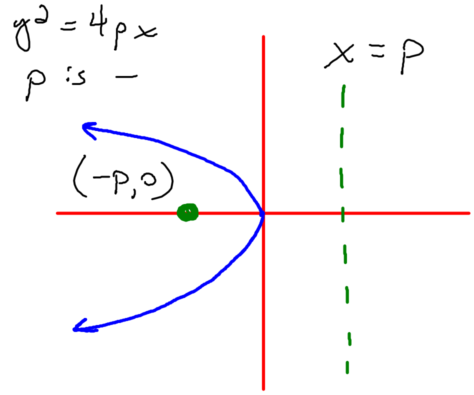


Quiz TUESDAY on 21.3 & 21.4


$x^2 = 4py$
p is +
(0, p)
$y = -p$


$x^2 = 4py$
p is −
$y = p$
(0, −p)


$x = -p$
$y^2 = 4px$
p is +
(p, 0)


$y^2 = 4px$
p is −
$x = p$
(−p, 0)